Economic growth theory

1 Introduction and empirical facts

- in **discrete time**, $x(t+s) = x(t)(1+\gamma_x)^s$ ($\implies$ compute average growth rate)
- in **continuous time**, $x(t+s) = x(t)e^{\gamma_x s}$. Further:
  - $\gamma_{xy}(t) = \gamma_x(t) + \gamma_y(t)$
  - $\gamma_{x/y}(t) = \gamma_x(t) - \gamma_y(t)$
  - $\gamma_{x^\alpha}(t) = \alpha\gamma_x(t)$
  - $\gamma_x(t) = \beta$ (constant) $\Leftrightarrow x(t) = Be^{\beta t}$
  - In a diagram with $\ln x(t)$ over t:
    - If $x(t)$ grows at constant rate $\beta, \ln x(t)$ shows up as a line with slope β.
    - If both $x(t)$ and $z(t)$ grow at the same (constant) rate, then $\ln x(t) - \ln z(t)$ remains constant $\implies$ the graphs are parallel lines.
- **Balanced growth**: Most economies typically exhibit balanced growth, i.e. they seem to follow a growth path along which the following variables are roughly constant:
  - the growth rates of real per-capita GDP, real per-capita consumption, and the per-capita capital stock
  - the real interest rate
  - the income shares of capital and labor

2 Basic assumptions, growth accounting

- Central assumptions:
  - Relative prices are flexible and all markets clear at all times.
  - Money is neutral and monetary variables can therefore be omitted from growth models.
- Aggregate production function: $Y(t) = \bar{F}(K(t), L(t), \bar{A}(t))$
- Aggregate model: a single output good is produced from two input factors (capital and labor) with a given technology
- changes in $\bar{A}(t)$ are interpreted as technological progress (or decline)
- Further assumptions
  - $\bar{F}$ (increasing, convave in K, L) is twice continuously differentiable in K, L
  - Positive marginal products of capital and labor: $\bar{F}_1(K, L, \bar{A}) > 0, \bar{F}_2(K, L, \bar{A}) > 0$
  - Diminishing marginal products (diminishing returns) of capital and labor: $\bar{F}_{11}(K, L, \bar{A}) < 0, \bar{F}_{22}(K, L, \bar{A}) < 0$
  - constant returns to scale (**linear homogeneity**) with respect to capital and labor: $\forall\lambda > 0, \forall(K, L, \bar{A}) \in \mathbb{R}^3_+, \bar{F}(\lambda K, \lambda L, \bar{A}) = \lambda\bar{F}(K, L, \bar{A})$ (no learning effects, no "crowding" effects)
- Factor prices under perfect competition (no monopolies, households and firms take price as given)
  - $w(t)$ denotes real wage at time t, $q(t)$ denotes real rental price of capital at time t (expressed in units of period-t output)

- Firms take the price of output, factor prices $w(t)$ and $q(t)$ and technology $\bar{A}(t)$ as given and maximize profits:
$$\max \bar{F}(K(t), L(t), \bar{A}(t)) - q(t)K(t) - w(t)L(t)$$
Thus,
$$q(t) = \bar{F}_1(K(t), L(t), \bar{A}(t)) \text{ and } w(t) = \bar{F}_2(K(t), L(t), \bar{A}(t))$$
- By Euler's theorem for linearly homogenous functions
$$\bar{F}(K, L, \bar{A}) = \bar{F}_1(K, L, \bar{A})K + \bar{F}_2(K, L, \bar{A})L$$
and therfore
$$Y(t) = q(t)K(t) + w(t)L(t)$$
- **income share of capital** is $\alpha(t) = q(t)K(t)/Y(t) = \bar{F}_1 K/\bar{F}$. This α is also the elastiticy of output w.r.t. capital. The income share of labor is $1 - \alpha(t)$.

- Capital accumulation equation
  - Output can be consumed or invested: $Y(t) = C(t) + I(t)$
  - Capital depreciates at a constant rate $\delta > 0$ (5-10 %). $I(t) = \delta K(t) + \dot{K}(t)$
  - Thus, $\dot{K} = I - \delta K = Y - C - \delta K = \bar{F} - C - \delta K$
- Harrod-neutral technological progress
  - Under the assumption that $\gamma_L(t) = n$ (constant population growth) [plus other assumptions, slide Ch.2-13], $\bar{F}(K(t), L(t), \bar{A}(t)) = F(K(t), A(t)L(t))$ and $\gamma_A(t) = \gamma_Y(t) - n$ (constant) [proof slide Ch.2-14]
  - Balanced growth is only possible if technological progress takes the Harrod-neutral (labor-augmenting) form $\bar{F}(K, L, \bar{A}) = F(K, AL)$ with a constant rate of technological progress $\gamma_A(t)$ (always assumed in this chapter)
  - $A(t)$ is referred to as **efficiency of labor** and $A(t)L(t)$ as **effective labor force**
- Production function in intensive form
  - intensive production function is defined by $f(k) = F(k, 1)$
  - f is continuous and twice continuously differentiable; $f'(k) > 0, f''(k) < 0 \ \forall k > 0$
  - Popular assumptions: $f(0) = 0$ (capital is essential) and $\lim_{k\to 0} f'(k) = +\infty, \lim_{k\to+\infty} f'(k) = 0$ (Inada conditions)
  - The graph of f is strictly increasing and strictly concave, starting at the origin.
  - $F_1(K, AL) = f'(k)$ (marginal productiviy of capital)
  - $F_2(K, AL) = f(k) - kf'(k)$ is strictly positive and strictly increasing w.r.t. k
  - $f(k)/k$ (average productivity of capital) is strictly decreasing
- Cobb-Douglas production function: $F(K, AL) = BK^\alpha(AL)^{1-\alpha}$, in intensive form $f(k) = Bk^\alpha$ (satisfies all assumptions and Inada conditions)
- Growth accounting: From $Y = F(K, AL)$ follows that $\gamma_{\tilde{y}} = \alpha\gamma_{\tilde{k}} + R$, where $\tilde{y} = Y/L, \tilde{k} = K/L, \alpha$ is the income share of capital and $R(t) = [1 - \alpha(t)]\gamma_A(t)$ is the **Solow residual** (measures the contribution of technological progress to economic growth)

3 The simplest neoclassical growth model (Solow-Swan model)

- Aggregate model; closed economy; exogenous population growth; exogenous technological progress; exogenous saving rate (no optimization)

$$
\begin{aligned}
&Y(t) = F(K(t), A(t)L(t)) \\
&\dot{A}(t) = gA(t), A(0) = A_0 \\
&\dot{L}(t) = nL(t), L(0) = L_0 \\
&Y(t) = C(t) + I(t) \\
&I(t) = \delta K(t) + \dot{K}(t) = sY(t), K(0) = K_0
\end{aligned}
$$

- Results
  - $A = A_0 e^{gt}, L = L_0 e^{nt}$
  - $k = K/(AL)$ (capital stock per unit of effective labor)
  - $\dot{k} = sf(k) - (n + \delta + g)k$
  - $\lim_{t \to +\infty} = k^*$, where k^* is determined uniquely by $sf(k^*) = (n + \delta + g)k^*$ and is called a fixed point or **steady state**
- Balanced growth path (all variables grow at constant rate)

$$
\begin{aligned}
&K = kAL = A_0 L_0 k^* e^{(n+g)t} \\
&Y = f(k)AL = A_0 L_0 f(k^*) e^{(n+g)t} \\
&\tilde{k} = K/L = A_0 k^* e^{gt} \\
&\tilde{y} = Y/L = A_0 f(k^*) e^{gt} \\
&\tilde{c} = C/L = (1 - s) A_0 f(k^*) e^{gt}
\end{aligned}
$$

- Long-run growth rate: $\lim_{t \to +\infty} \gamma_{\tilde{y}}(t) = g$ (the only exogenous variable that affects the long-run growth rate is the rate of technological progress g)
- Long-run values of capital, output and consumption [graphics in notes, Ch.3-23]
  - $sf(k^*) = (n + \delta + g)k^* \implies k^*$ is increasing in s and decreasing in n, δ, g (analogously for y^*)
  - $c^* = (1 - s)f(k^*) = f(k^*) - sf(k^*) = f(k^*) - (n + \delta + g)k^*$

$$
\frac{dc^*}{ds} \begin{cases} > 0 & \text{if } k^* < \bar{k} \\ = 0 & \text{if } k^* = \bar{k} \\ < 0 & \text{if } k^* > \bar{k} \end{cases}
$$

  where $\bar{k}$ is uniquely determined by $f'(\bar{k}) = n + \delta + g$
- The Golden Rule of dynamic inefficiency
  - $\bar{k}$ is the **Golden-Rule capital stock** (k for which steady state consumption per unit of effective labor is maximized)
  - $\bar{k}$ is a steady state if and only if saving rate $\bar{s} = (n + \delta + g)\bar{k}/f(\bar{k})$ (Golden-Rule saving rate; the saving rate which implies the highest long-run consumption per unit of effective labor)
  - If $s < \bar{s}$, then a small increase of s leads to a reduction of consumption for the present generation and to an increase of consumption for future generations (trade-off)
  - If $s > \bar{s}$, then a small reduction of s leads to an increase of consumption for all generations (Pareto-improvement)

- A saving rate $s > \bar{s}$ is called **dynamically inefficient** (capital overaccumulation)
- Empirically, typically $s < \bar{s}$

- The Cobb-Douglas case
  - $F(K, AL) = BK^\alpha (AL)^{1-\alpha} \implies \dot{k}(t) = sBk(t)^\alpha - (n + \delta + g)k(t)$
  - Steady state (where $\dot{k} = 0$) can be computed explicitly [Ch.3-25]
- Transition dynamics in the general case [Ch.3-26]: a first-order approximation for the rate of convergence close to the fixed point k^* is given by $\lambda = (1 - \alpha^*)(n + \delta + g)$
- Major predictions of the Solow-Swan model
  - Technological progress is necessary for long-run growth. Capital accumulation alone cannot sustain growth due to diminishing returns to capital.
  - The model predicts conditional convergence of growth rates and per-capita income levels (if technological progress is the same in two countries, also their long-run growth rates should be the same; if additionally their $f, n + \delta + g$ are the same, then also their long-run per-capita income levels should be the same)
- Elasticity of long-run output level w.r.t. the saving rate is $\frac{s}{y^*}\frac{\partial y^*}{\partial s} = \frac{\alpha^*}{1-\alpha^*}$. Along a balanced growth path, also the elasticity of per-capita GDP w.r.t. the saving rate is $\frac{\alpha^*}{1-\alpha^*}$, which would result in an unreasonably small dispersion among countries.

4 Cross-country income differences and convergence

- **absolute β convergence**: "Poorer" country grows faster unconditionally (only supported for "similar" countries like OECD countries; not supported by empirical evidence if the cross-section of countries is sufficiently broad, Ch.4-31).
- **conditional β convergence**: "Poorer" country grows faster only if it is farther away from its balanced growth path.
- **σ convergence**: The dispersion of per-capita income levels across countries relative to the average income level declines over time (coefficient of variation).

5 Traditional neoclassical growth theories

(a) Infinitely-lived households (Ramsey-Cass-Koopmans model)

- Dynamic general-equilibrium model: goods market, labor market, capital market
- Walrasian model: perfect competition, rational expectations (perfect foresight), complete markets
- Explicit micro foundations: firms maximize profits, households maximize utility; households own the factors of production; welfare analysis is possible
- The saving rate is endogenously determined as part of the solution of the households' optimization problem
- Firms
  - Unit interval of identical firms (aggregate output, employment etc. equals output, employment etc. of the representative firm)
  - The representative firm produces output at time t according to the production function $Y(t) = F(K(t), A(t)L(t))$

  - The firm takes capital rent $q(t)$, real wage rent $w(t)$ and the state of technology $A(t)$ as given and maximizes profits w.r.t. $Y(t), K(t)$ and $L(t)$ (static optimization in each period; NO dynamic optimization).
  - FOC: $q(t) = F_1(K, AL), w(t) = AF_2(K, AL)$
  - $y = Y/(AL), k = K/(AL) \implies$ conditions in intensive form:

$$
\begin{aligned}
y(t) &= f(k(t)) \\
q(t) &= f'(k(t)) \text{ marginal productivity of capital} \\
w(t) &= A(t)[f(k(t)) - k(t)f'(k(t))]
\end{aligned}
$$

- Technology: $\dot{A} = gA(t)$, capital depreciates at constant rate $\delta \implies$ return to capital (real interest rate) $r(t) = q(t) - \delta$
- The lifetime budget constraint of the households
  - H identical and infinitely-lived households (dynasties), each growing at constant rate n. Population size (labor force) L satisfies $\dot{L} = nL$
  - Every member of households is endowed with one unit of time to work $\implies$ labor supply of single household is $L(t)/H$
  - Every household owns at time t capital stock $K(t)/H$, where $K(0)$ is a given constant.
  - Every individual consumes at time t the amount $\tilde{c}(t) = C(t)/L(t)$, i.e. every household consumes $C(t)/H$
  - Lifetime budget constraint: the present value of a household's consumption between time 0 and $+\infty$ must not exceed the present value of its endowment with capital and total wage income:

$$\int_{t=0}^{+\infty} e^{-R(t)} C(t)/H dt \leq K(0)/H + \int_{t=0}^{+\infty} e^{-R(t)} w(t)L(t)/H dt$$

    where the first term is the discount factor, the second the consumption expenditure in t, the third the initial endowment, and the forth the present value of wage income of the entire lifetime.

    Capital income is included in R:

$$R(t) = \int_0^t r(\tau) d\tau$$

- Present value of capital stock at time T is

$$e^{-R(T)} K(T) = K(0) + \int_{t=0}^{T} e^{-R(t)} [w(t)L(t) - C(t)] dt$$

  which upon differentiation w.r.t T, multiplying by $e^{R(t)}$ and using $\dot{R}(T) = r(T)$ yields

$$\dot{K}(t) = w(t)L(t) + r(t)K(t) - C(t)$$

- Transversality condition: $\lim_{t\to+\infty} e^{-R(t)} K(t) = 0$ (at the end of time, no capital should be left back)
- In per-capita variables, $\dot{\tilde{k}} = w(t) + [r(t) - n]\tilde{k}(t) - \tilde{c}(t)$
- In intensive variables, $\dot{k} = w(t)/A(t) + [r(t) - (n+g)]k(t) - c(t)$
- Households maximize $\int_{t=0}^{+\infty} = e^{-\rho t} U(\tilde{c}(t)) L(t)/H dt$, where ρ measures relative weight given to future consumption (high ρ corresponds with high impatience)

- As $\gamma_L(t) = n$, the objective function can be rewritten as $\int_{t=0}^{+\infty} e^{-(\rho-n)t} U(\tilde{c}(t))dt$
- So, the households maximizes

$$\int_{t=0}^{+\infty} e^{-(\rho-n)t} U(\tilde{c}(t))dt$$

subject to

$$\begin{aligned}
&\dot{\tilde{k}} = w(t) + [r(t) - n]\tilde{k}(t) - \tilde{c}(t) \\
&\lim_{t\to+\infty} e^{-R(t)} e^{nt} \tilde{k}(t) \geq 0 \\
&\tilde{c}(t) \geq 0
\end{aligned}$$

choosing $\tilde{k}(t), \tilde{c}(t)$ and $w(t), r(t)$ given

- Euler equation [Ch.5a-41, notes]: $\gamma_{\tilde{c}} = \frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} = \frac{r(t)-\rho}{\epsilon_U(\tilde{c}(t))}$, where ϵ_U is the elasticity of the marginal utility at $\tilde{c}$ or the inverse of the elasticity of intertemporal substitution: $\epsilon_U(\tilde{c}) = -\frac{U''(\tilde{c})\tilde{c}}{U'(\tilde{c})} > 0$
- If $r(t) > \rho$, it pays to postpone consumption, i.e., current per-capita consumption must be **smaller** than future consumption ($\gamma_{\tilde{c}} > 0$)
- If $\rho > r(t)$, the household is very impatient and saves only little $\implies$ present per-capita household must be **higher** than future consumption ($\gamma_{\tilde{c}} < 0$)
- These effects are strong, if the **elasticity of intertemporal substitution** is large
- **Balanced growth and the elasticity of substitution**: By the Euler equation, there is only a constant growth rate (balanced growth path) of per-capita consumption if depending on the constant real interest rate there exists a constant $\theta > 0$ s.t. $\epsilon_U(\tilde{c}) = \theta$ holds for all $\tilde{c} \geq 0$. Thus

$$U(\tilde{c}) = \begin{cases} \frac{\tilde{c}^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1 \\ \ln(\tilde{c}) & \text{if } \theta = 1 \end{cases}$$

where θ determines the curvature of the utility function. The higher the curvature, the stronger is the desire of the household to have a smooth consumption stream $\implies$ θ determines how strongly a household reacts to intertemporal price differences. The **elasticity of intertemporal substitution** is equal to $1/\theta$.

- **Equilibrium**

Firms:

$$\begin{aligned}
&r(t) = f'(k(t)) - \delta \\
&w(t) = A(t)[f(k(t)) - k(t)f'(k(t))]
\end{aligned}$$

Households:

$$\begin{aligned}
&\dot{k}(t) = w(t)/A(t) - [r(t) - n - g]k(t) - c(t) \\
&\dot{c}(t) = c(t)[r(t) - \rho - \theta g]/\theta \text{ from Euler equation} \\
&k(0) = k_0 = K(0)/[A(0)L(0)] \\
&\lim_{t\to+\infty} e^{-R(t)} e^{(n+g)t} k(t) = 0
\end{aligned}$$

Equilibrium:

$$\begin{aligned}
&\dot{k}(t) = f(k(t)) - (n + g + \delta)k(t) - c(t) \\
&\dot{c}(t) = c(t)[f'(k(t)) - \rho - \delta - \theta g]/\theta
\end{aligned}$$

subject to boundary conditions:

$$k(0) = k_0$$
$$\lim_{t \to +\infty} e^{-R(t)} e^{(n+g)t} k(t) = 0$$

- Phase diagram analysis [Ch.5a-45, notes]
  - The isocline $\dot{c}(t) = 0$ consists of the two lines $c = 0$ and $k = k^*$, where k^* is determined by $f'(k^*) = \rho + \delta + \theta g$ (so $c(t)$ grows to the left of this line, and falls to its right)
  - The isocline $\dot{k}(t) = 0$ is the curve $c = f(k) - (n + \delta + g)k$ (so $k(t)$ grows below this curve and falls above it)
  - The isocline $\dot{k}(t) = 0$ attains its maximum at $k = \bar{k}$, where $\bar{k}$ is the Golden-rule capital stock. Generally, $k^* < \bar{k}$
  - Three fixed points of the system of two differential equations: $(0, 0), (k^*, c^*), (\hat{k}, 0)$
- Stable saddle point path
  - In equilibrium, the differential for $k(t)$ and $c(t)$ and the initial and transversality conditions must be satisfied.
  - If $c(0)$ large, $k(t)$ becomes negative, so this cannot be a solution
  - If $c(0)$ small, solutions converge to $(\hat{k}, 0)$, in which case the transversality condition would be violated, so this cannot be a stable saddle point path either.
  - Only for one unique value $c(0)$ the solutions remain feasible and satisfy initial and transversality conditions. This solution converges to (k^*, c^*) (increasing curve in (k, c) diagram)
- Balanced growth path
  - As $k(t) \to k^*$, it follows that $y(t) \to y^* = f(k^*)$ and $c(t) \to c^* = f(k^*) - (n + \delta + g)k^*$
  - As in the Solow-Swan model, all equilibria converge to a balanced growth path

$$K = kAL = A_0 L_0 k^* e^{(n+g)t}$$
$$Y = f(k)AL = A_0 L_0 f(k^*) e^{(n+g)t}$$
$$\tilde{k} = K/L = A_0 k^* e^{gt}$$
$$\tilde{y} = Y/L = A_0 f(k^*) e^{gt}$$
$$\tilde{c} = C/L = A_0 [f(k^*) - (n + \delta + g)k^*] e^{gt}$$

  - The long-run growth rate is g. No other parameter has a long-run growth effect.
  - k^* is a decreasing function of ρ, δ, θ and g, and is independent of n
  - c^* is a decreasing function of $\rho, \delta, \theta, g, n$.
- **Transition dynamics** - non-anticipated parameter change at time t_0 [notes]
  - Before t_0, the equilibrium is described by the old stable saddle point path. Immediately before t_0, the economy is in state $(k(t_0^-), c(t_0^-))$.
  - There is a new stable saddle point path corresponding to the new parameters.
  - After t_0, the equilibrium follows the new saddle point path starting in $k(t_0) = k(t_0^-)$
- **Transition dynamics** - parameter change at time t_1 that becomes already known at time t_0 (announcement effect)

  - Before t_0 the equilibrium is described by the old stable saddle point path. Immediately before t_0, the economy is in state $(k(t_0^-), c(t_0^-))$.
  - There is a new stable saddle point path corresponding to the new parameters.
  - After t_1 the equilibrium follow the new saddle point path.
  - Between t_0 and t_1, the equilibrium is characterized by that solution of the **old phase diagram**, which leads from $k(t_0) = k(t_0^-)$ in exactly $t_1 - t_0$ time units to the new stable saddle point path.

- **Adjustment speed**: Linearization around the fixed point [Ch.5a-49; notes] with some reasonable parameters yields $\lambda_1 = -12.1\%, \lambda_2 = 15.1\%$ (which can be disregarded on economic grounds?), i.e., within 6 years the distance between $k(t)$ and k^* is reduced by 50% (faster convergence than in the Solow model).
- For $g = 0, d\lambda_1/d\theta > 0$, i.e. higher elasticity of intertemporal substitution leads to higher adjustment speed.
- **Pareto-efficiency**: A social planner has to obey technological constraints and economy-wide resource constraints, but is not subject to budget constraints. His FOC are identical to decentralized economy. Thus, the market equilibrium obtained before is **Pareto-efficient** (because there is a "perfect" market, and everybody is perfectly rational). The **First Welfare Theorem** applies, i.e., there is no role for the government in this model.

(b) Overlapping generations of finitely-lived households (Diamond model)

- Similarities to Ramsey-Cass-Koopmans model:
  - Households and firms interact on the markets for goods, capital, labor (dynamic general equilibrium model).
  - Perfect competition, complete markets, perfect foresight (Walrasian model).
  - Saving rate endogenously determined by the households' utility maximization problem.
- Main differences to Ramsey-Cass-Koopmans model:
  - Households have **finite** lifetimes, but there are **infinitely** many overlapping generations of households.
  - The structure of the equilibrium set and the equilibrium dynamics can be more complicated
  - Equilibria need not be Pareto-efficient (First Welfare Theorem fails!)
  - Technical assumption: Discrete time periods. Every household lives for two periods.
- Production and technological progress as in the Ramsey-Cass-Koopmans model: A representative firm which takes rental rate of capital and wage rate as given and (statically) maximizes its profit. Constant rate of depreciation δ and constant rate of technological progress g.

$$
\begin{aligned}
y(t) &= f(k(t)) \\
q(t) &= f'(k(t)) \\
w(t) &= A(t)[f(k(t)) - k(t)f'(k(t))] \\
r(t) &= q(t) - \delta \\
A(t+1) &= (1+g)A(t)
\end{aligned}
$$

- **Budget constraint of the households**
  - At the start of period t, $L(t)$ identical households are born and live until $t+1$, $L(t+1) = (1+n)L(t)$. Every household has one member.

- Every households works in its first period of life, but do not work in the second period. It consumes in both periods of life.
- $C_t(t) + \frac{C_t(t+1)}{1+r(t+1)} = w(t)$ (total lifetime labor income)
- Equivalently, $C_t(t) = (1-s_t)w(t)$ and $C(t+1) = [1+r(t+1)]s_t w(t)$, where $s_t = [w(t)-C_t(t)]/w(t)$ denotes the saving rate of the household (in first period of life, there is no bequest).

- **Utility maximization of the households**:

$$\max U(C_t(t)) + \beta U(C_t(t+1))$$

s.t. budget constraint, where $\beta > 0$ is the weight given to old-age utility relative to young-age utility (**time preference factor**).

- FOC (Euler Equation):

$$U'(C_t(t)) = \beta[1 + r(t+1)]U'(C_t(t+1))$$

, which can be solved for $C_t(t), C_t(t+1)$ or for s_t as functions of $w(t), r(t+1)$, i.e, we obtain $s_t = S(w(t), r(t+1))$.

- Capital accumulation:

$$K(t+1) = s_t w(t)L(t) = S(w(t), r(t+1))w(t)L(t)$$

or equivalently

$$k(t+1) = \frac{S(w(t), r(t+1))w(t)}{(1+n)A(t+1)}$$

which together with FOC for profit maximization yields

$$k(t+1) = \frac{S(A(t)W(k(t)), R(k(t+1)))W(k(t))}{(1+n)(1+g)}$$

where $W(k) = f(k) - f'(k)k$ and $R(k) = f'(k) - \delta$.

- Every sequence $(k(t))_{t=0}^{+\infty}$ which satisfies this difference equation as well as the initial condition $k(0)$ is an **equilibrium**. A constant sequence (fixed point of the difference equation) corresponds to a **balanced growth path**. For a balanced growth path to exist, the explicit dependence on $A(t)$ should vanish.

- CES utility function: Assume U is given by

$$U(C) = \begin{cases} \frac{C^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1 \\ \ln C & \text{if } \theta = 1 \end{cases}$$

then we obtain

$$S(w, r) = \frac{\beta^{1/\theta}(1+r)^{(1-\theta)/\theta}}{1 + \beta^{1/\theta}(1+r)^{(1-\theta)/\theta}}$$

In this case, the saving rate is independent of the wage rate and increasing or decreasing w.r.t. the interest rate r depending on whether the substitution effect dominates ($\theta < 1$) or the income effect dominates ($\theta > 1$). If $\theta = 1, S(w, r) = \beta/(1+\beta)$, i.e., the saving rate is constant.

- Cobb-Douglas technology and logarithmic utility: Suppose U has CES with $\theta = 1$ and $f(k) = Bk^\alpha$. Then $S(w, r) = \beta/(1+\beta), W(k) = (1-\alpha)Bk^\alpha, R(k) = \alpha Bk^{1-\alpha} - \delta$.

$$k(t+1) = \frac{(1-\alpha)\beta Bk(t)^\alpha}{(1+n)(1+g)(1+\beta)}$$

- Graphically [notes] we obtain a unique balanced growth path which is globally stable:

$$k^* = \left[\frac{(1-\alpha)\beta B}{(1+n)(1+g)(1+\beta)}\right]^{1/(1-\alpha)}$$

- Along the balanced growth path, per-capita GDP grows at the rate of technological progress, i.e., $k(t) = k^* A(t) L(T)$
- The fixed point k^* is increasing in B and β and decreasing in n and g.
- Adjustment speed: Within one generation, the distance to the steady state is reduced by two thirds.

- The general case (neither CES utility nor Cobb-Douglas technology), the equilibrium conditions allow for more complicated solutions:
  - Multiple balanced growth paths. The initial capital endowment determines to which balanced growth path the economy converges. Sensitivity with respect to initial conditions. History-dependence. Poverty traps.
  - There may exist multiple values of $k(t+1)$ which are consistent with the equilibrium condition and a given value of $k(t)$. Possibility of multiple equilibria and sunspot equilibria (self-fulfilling prophecies). Extrinsic uncertainty can affect the equilibria (in line with rationality; Keynes: "animal spirits")
- **Welfare analysis**: Equilibria can be inefficient, i.e., a social planner may be able to improve upon the market equilibrium (Pareto-improvement). E.g. when young generation saves to much, the introduction of a pay-as-you-go pension system can allow all households to consume more in all periods [Ch.5b-59].
- In traditional neoclassical growth theories (Solow-Swan, Ramsey-Cass-Koopmans, Diamond model), the long-run growth rate of the economy is determined solely by the rate of technological progress (which is exogenous). Capital accumulation alone cannot lead to long-run growth.
- Neoclassical growth theories cannot explain the high dispersion of growth rates and per-capita GDP levels across countries.

6 "New" growth theories

(a) Human capital and the cross-country income distribution

- Human capital = acquired skills (embodied to person, 100% depreciation when you die).
- Cobb-Douglas production function with three input factors:

$$\begin{aligned}
Y(t) &= K(t)^\alpha H(t)^\beta [A(t)L(t)]^{1-\alpha-\beta} \\
\dot{L}(t) &= nL(t) \\
\dot{A}(t) &= gA(t) \\
\dot{K}(t) &= s_K Y(t) - \delta_K K(t) \\
\dot{H}(t) &= s_H Y(t) - \delta_H H(t)
\end{aligned}$$

- Standard Solow-Swan model is a special case of this model with $\beta = 0$
- Exogenous technological progress.
- $y(t) = k(t)^\alpha h(t)^\beta$, so

- $\gamma_k(t) = \gamma_K(t) - \gamma_A(t) - \gamma_L(t) = s_K \frac{y(t)}{k(t)} - \delta_K - g - n$ and therefore
- $\dot{k}(t) = s_K k(t)^\alpha h(t)^\beta - (n + g + \delta_K)k(t)$ and
- $\dot{h}(t) = s_H k(t)^\alpha h(t)^\beta - (n + g + \delta_K)h(t)$
- Phase diagram analysis shows that all solutions converge to the fixed point (k^*, h^*) [Ch.6a-63; notes]
- Elasticities: Assuming $\delta_K = \delta_H = \delta$, the elasticity of long-run per-capita output w.r.t. the saving rate s_K is

$$\frac{d\tilde{y}^*/\tilde{y}^*}{ds_K/s_K} = \frac{d\ln\tilde{y}^*}{d\ln s_K} = \frac{\alpha}{1-\alpha-\beta}$$

  and

$$\frac{d\tilde{y}^*/\tilde{y}^*}{d(n+g+\delta)/(n+g+\delta)} = -\frac{\alpha+\beta}{1-\alpha-\beta}$$

- If $\beta > 0$ instead of $\beta = 0$, the elasticities are larger, and therefore small differences in the saving rate and the population growth rate can explain larger differences in per-capita income levels.
- β is that fraction of income that is earned by human capital ("income share of skills")
- According to an empirical estimation by Mankiw, Romer and Weil (1992), the standard Solow-Swan model cannot explain the data as well as the variant including human capital.

(b) Returns to scale and the AK-model

- In traditional neoclassical growth models without technological progress, long-run growth of per-capita GDP is ruled out by diminishing maringal returns of capital (second Inada condition).
- E.g., $\tilde{y} = A\tilde{k}(t)^\alpha$ with $A > 0, \alpha \in (0,1)$. Then $\gamma_{\tilde{y}}(t) = \alpha\gamma_{\tilde{k}}(t) < \gamma_{\tilde{k}}(t)$. In the long-run, the income generated in the economy is too small to finance the capital stock.
- Without exogenous technological progress, a balanced growth path with a positive growth rate of per-capita GDP can only exist if the technology F has non-decreasing returns to scale with respect to the accumulable factors of production (e.g., physical capital, human capital).
- This is only possible if
  - (i) F has constant returns to scale and all production factors are accumulable, e.g., $F(K, H, S, \ldots) = AK^{\alpha_1}H^{\alpha_2}S^{1-\alpha_1-\alpha_2} \implies AK$-model (constant returns to scale, but all factors are accumulable)
  - (ii) F has increasing returns to scale, e.g., $F(K, H, S, \ldots, L) = AK^{\alpha_1}H^{\alpha_2}S^{1-\alpha_1-\alpha_2}L^\beta$ (increasing returns to scale)
- AK-model: Ramsey-Cass-Koopmans model with a single production factor (capital) and constant returns to scale

$$\dot{L}(t) = nL(t), Y(t) = AK(t), g = 0$$
$$w(t) = 0, r(t) = A - \delta, R(t) = (A-\delta)t$$

  As $g = 0$, there is no exogenous technological progress!

- Equilibrium conditions in per-capita terms:

$$\dot{k}(t) = Ak(t) - (n+\delta)k(t) - c(t)$$
$$\dot{c}(t) = c(t)(A - \delta - \rho)/\theta$$
$$k(0) = k_0$$
$$\lim_{t\to+\infty} e^{-(A-\delta-n)t}k(t) = 0$$

- If $A > \delta + \rho$, there will be positive growth of per-capita consumption.
- If $A-(n+\delta) > (A-\delta-\rho)/\theta$, then for every k_0 there is an unique solution, which is a balanced growth path with $\gamma_k(t) = \gamma_c(t) = \gamma_y(t) = (A-\delta-\rho)/\theta > 0$ (there is no convergence among countries; the economy is always on the balanced growth path)
- Implications of the AK-model
  - No transitional dynamics; every solution of the model is a balanced growth path.
  - The model does not predict convergence of per-capita GDP levels (not even conditional β convergence; there can be even difergence if $\rho_1 > \rho_2$
  - The growth rate depends on policy, i.e., economies with different tax rates that are otherwise identical exhibit diverging growth paths.
  - An income share of capital of 1 (i.e., the wage rate is 0) and unstable cross-country distributions of per-capita income levels are unrealistic.

(c) Learning-by-doing and knowledge spillover

- Examples of knowledge: scientific results, product designs, production processes (contrary to human capital, knowledge is **not** embodied).
- Non-rival good, but often excludable good (secrecy, patents).
- Knowledge increases both through learning-by-doing (external effect of production) and research and development, incentivized through future profits (market power through imperfect competition is essential).
- Increasing returns to scale on the **firm level** are neither consistent with the replication argument nor with perfect competition.
- Thus, in this model, there is a continuum of identical firms $i \in [0,1]$ with **individual** production function
$$Y_i(t) = F(K_i(t), A(t)L_i(t)) = K_i(t)^\alpha [A(t)L_i(t)]^{1-\alpha}, \alpha \in (0,1)$$
So there are constant returns to scale w.r.t $K_i(t), L_i(t)$.
- External effect: Every firm takes the labor efficiency $A(t)$ as given, because they cannot affect $A(t)$. However, the actions of all firms together do determine the evolution of $A(t)$!
- Non-rivalness of knowledge.
- Due to the external affect, the aggregate production function is different from the individual one!
- Learning-by-doing: efficiency of labor is increased, but this effect is not taken into account by firms when they make their investment decision. Thus, the efficiency of labor is an increasing function of the **aggregate** capital stock:
$$A(t) = BK(t)$$
with $B > 0$
- Perfect competition and identical firms (i.e., $K_i(t) = K(t), L_i(t) = L(t)$
$$r(t) = F_1(K_i(t), A(t)L_i(t)) - \delta = \alpha[BL(t)]^{1-\alpha} - \delta$$
$$w(t) = A(t)F_2(K_i(t), A(t)L_i(t)) = (1-\alpha)K(t)B^{1-\alpha}L(t)^{-\alpha}$$
- The aggregate production function:
$$Y(t) = K(t)^\alpha [A(t)L(t)]^{1-\alpha} = K(t)[BL(t)]^{1-\alpha}$$
which exhibits **increasing returns to scale** w.r.t. $K(t), L(t)$!

- Supposing $L(t) = L$, the optimality conditions for the households are

$$\dot{\tilde{k}} = w(t) + r(t) + \tilde{k} - \tilde{c} = (BL)^{1-\alpha}\tilde{k}(t) - \delta\tilde{k}(t) - \tilde{c}(t)$$
$$\dot{\tilde{c}}(t) = \tilde{c}(t)[r(t) - \rho]/\theta = \tilde{c}(t)[\alpha(BL)^{1-\alpha} - \delta - \rho]/\theta$$

- As in the AK-model, under certain conditions, for every $\tilde{k}_0$ there exists a unique solution and this solution is a balanced growth path.
- The growth rate depends on the size of the economy L (scale effect).
- If there were population growth $n > 0$, this model would predict **explosive growth**. A model with weaker spillover $A(t) = BK(t)^{\epsilon}$ with $\epsilon < 1$ also works for $n > 0$.
- **Social planner solution**: Because of the external effect, the decentralized equilibrium is not Pareto optimal. A social planner would optimize the households' utility function

$$\int_{t=0}^{+\infty} e^{-\rho t}[\tilde{c}(t)^{1-\theta} - 1]/(1-\theta)dt$$

s.t. economy-wide resource constraint

$$\dot{\tilde{k}}(t) = \tilde{y}(t) - \tilde{c}(t) - \delta\tilde{k}(t)$$

where $\tilde{y}(t) = (BL)^{1-\alpha}\tilde{k}(t)$

- The Pareto-optimal growth rates on the balanced growth path is **always larger** than the equilibrium growth rate, because the social planner takes into account that investing new capital improves the technology. There is **underinvestment** in the competitive equilibrium. Due to the external effect, the First Welfare Theorem is violated.

(d) Research and development: the main ideas

- What are the incentives for process innovation in a partial equilibrium setting with an industry demand curve $X = D(p)$?
- There is a large number of firms in the industry. The existing technology has constant marginal cost w (e.g. wage rate).
- Each firm can reduce the marginal cost to w/λ at a fixed cost $\mu > 0$ ($\lambda > 1$).
- Without any innovation, due to perfect competition $p = w$ and all firms make zero profit.
- If a firm innovates and the new knowledge is non-excludable, the new equilibrium price will be $p = w/\lambda$, all firms except the innovator will make zero profits, and the innovator will have negative profit $-\mu$.
- Because of non-rivalness and non-excludability, the innovator cannot appropriate the gains for the innovation. There are **no incentives to innovate**.
- The **social value of innovation** is

$$S = \int_{w/\lambda}^{w} D(p)dp - \mu = \int_{w/\lambda}^{w} [D(p) - D(w)]dp + D(w)[w - (w/\lambda)] - \mu$$

where the first term captures the increase in consumer surplus due to **new customers** attracted by the lower price, the second term reflects the **increased surplus for the existing customers**, and the last term is the **cost of the innovation**.

- Still, innovation will only happen when there is temporary monopoly power for the innovator, he can keep his innovation secret of he can protect his rights by a patent.

- **Patents**: Under a fully enforced and perpetual patent, the innovator is able to undercut its competitors and **capture the entire market** (i.e., he gains ex-post **monopoly power**). This may create incentives for innovation!
- Assume $D(p) = p^{-\epsilon}$ with elasticity $\epsilon > 1$.
- **Drastic innovation**: $\lambda \geq \epsilon/(\epsilon-1)$. A monopolist with marginal cost w/λ chooses p so as to maximize $D(p)[p-(w/\lambda)]$, yielding the monopoly price

  $$p_M \frac{\epsilon}{\epsilon-1}\frac{w}{\lambda}$$

  satisfying $w/\lambda < p_M \leq w$. The monopoly profit is

  $$\pi_M(\mu) = D(p_M)[p_M - (w/\lambda)] - \mu$$

- **Non-drastic innovation**: $\lambda < \epsilon/(\epsilon-1)$. As the monopoly price p_M is now larger than w, setting $p = p_M$ does not attract any demand. However, the innovator can still capture the entire market by charging a price just below the competitive one. Thus, the innovator uses the **limit price** ("one cent less")

  $$p_L = w$$

  Then, the innovators net profit is

  $$\pi_L(\mu) = D(w)[w - (w/\lambda)] - \mu$$

- The social value of the innovation and the creation of a monopoly is

  $$S_M = D(p_M)[p_M - (w/\lambda)] + \int_{p_M}^{w} D(p)dp - \mu = \pi_M(\mu) + \int_{p_M}^{w} D(p)dp$$

  respectively

  $$S_L = D(w)[w - (w/\lambda)] - \mu = \pi_L(\mu)$$

  .
- It holds that $\pi_M(\mu) < S_M < S$ and $\pi_L(\mu) = S_L < S$
- The social planner is at least as willing to adopt an innovation as the individual firm, because a monopolistic firm can only appropriate part of the additional consumer surplus generated by the innovation.
- **Replacement effect**: Suppose there is already an unrestricted monopolist producing at marginal cost w, setting the price $\bar{p}_M = w\epsilon/(\epsilon-1)$ and making profit $\bar{\pi}_M = D(\bar{p}_M)(\bar{p}_M - w)$.
- If the monopolist makes the innovation, it remains a monopolist, faces marginal cost w/λ, charges p_m and makes profit $\pi_M(\mu)$. The value of the innovation to the monopolist is then

  $$\Delta\pi = \pi_M(\mu) - \bar{\pi}_M$$

  which can be both positive or negative, depending on μ and ϵ.
- If not the monopolist (incumbent) makes the innovation, but a competing firm (entrant), then the value to that firm is $\pi_M(\mu)$ (drastic innovation).
- Generally,

  $$\Delta\pi_M < \pi_M(\mu)$$

  So the incumbent has a **weaker incentive** to innovate than an entrant, because the innovation will replace its own already existing profits (Schumpeter's creative destruction).

Business cycle theory

1. Introduction and stylized facts

- Business cycle are short-run fluctuations of macroeconomic variables around their long-run trends.
- Trend and cycle component: $Y_t = \bar{Y}_t \tilde{Y}_t$, or in logarithmic terms $y_t = \bar{y}_t + \tilde{y}_t$
- **Hodrick-Prescott filter**: trend $(\bar{y}_t)_{t=1}^T$ is defined as the solution of the minimization problem

$$\sum_{t=1}^{T} (y_t - \bar{y}_t)^2 + \lambda \sum_{t=2}^{T-1} [(\bar{y}_{t+1} - \bar{y}_t) - (\bar{y}_t - \bar{y}_{t-1})]^2 \to min$$

  where λ is a relative weight to keep the trend smooth.

2. Real business cycle theory

(a) The standard model

- Main assumptions:
  - Walrasian models (complete markets, perfect competition, no external effects, no asymmetric information)
  - Money is neutral (change in the money supply affect nominal variables but they do not affect real variables). No nominal variables.
  - Real demand and supply shocks are the only causes of the business cycle
- standard RBC model (Kydland and Prescott 1982, Long and Plosser 1983) is similar to the Ramsey-Cass-Koopmands model, but with three major differences: stochastic technology shocks, endogenous labor supply, discrete time formulation.
- Cobb-Douglas production function: $Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}, 0 < \alpha < 1$
- $\ln A_t = \ln A_0 + gt + \tilde{a}_t$, whereby the productivity shocks $(\tilde{a}_t)_{t=0}^{+\infty}$ are an AR(1) process:

$$\tilde{a}_t = \rho_A \tilde{a}_{t-1} + \epsilon_t$$

- $-1 < \rho_A < 1$ (autocorrelation coefficient). ϵ_t is white noise (i.i.d. with $E(\epsilon_t) = 0$).
- Real interest rate: $r_t = \alpha K_t^{\alpha-1} (A_t L_t)^{1-\alpha} - \delta = \alpha Y_t / K_t - \delta$
- Real wage: $w_t = (1-\alpha) K_t^\alpha A_t^{1-\alpha} L_t {-} \alpha = (1-\alpha) Y_t / L_t$
- Population in t is N_t, growing at constant n, i.e., $N_t = N_0 e^{nt}$ or $\ln N_t = \ln N_0 + nt$
- Aggregate labor supply in t is L_t. Per-capita leisure ("unemployment rate") is $1 - L_t/N_t$ (endogenous labor supply decision)
- Instantaneous utility of individual in t is $U(C_t/N_t, 1 - L_t/N_t)$, and time-preference rate ρ
- Simplifying assumption: $U(C/N, 1 - L/N) = \ln(C/N) + b \ln(1 - L/N)$, where $b > 0, n < \rho$ and $\theta = 1$ (level of risk aversion)
- **Social planner's problem**

$$E_0 \{ \sum_{t=0}^{+\infty} e^{-\rho t} N_t U(C_t/N_t, 1 - L_t/N_t) \}$$

s.t. technological constraint

$$K_{t+1} = (1-\delta)K_t + K_t^\alpha (A_t L_t)^{1-\alpha} - C_t$$

- exogenously given productivity shocks $\implies$ stochastic optimization problem $\implies$ decision maker maximizes expected utility w.r.t information available in t, choosing C_t (resp. K_{t+1}) and L_t
- Optimality conditions c.f. Ch.2a-10, notes
- FOC for L_t and K_{t+1} yield stochastic Euler equation:

$$e^{-(\rho-n)} E_t \frac{1+r_{t+1}}{C_{t+1}} = \frac{1}{C_t}$$

If ρ is large relative to real interest rate r_{t+1}, then it is optimal to choose current consumption higher then expected future consumption. If it is small, choose an increasing consumption path.

- Intertemporal substitution of labor (Ch.2a-12, notes):

$$\frac{1}{N_t - L_t} = e^{-(\rho-n)} E_t \left[\frac{w_t}{w_{t+1}} \frac{1+r_{t+1}}{N_{t+1} - Lt + 1} \right]$$

- The more real wage is expected to increase from t to $t+1$, the smaller L_t should be chosen.
- The higher the expected real interest rate for $t+1$, the higher L_t should be chosen.
- Shocks affect productivity and thereby factor prices, which in turn affects employment.

(b) A special case

- General RBC model cannot be solved analytically, but the special case with $\delta = 1$ (full depreciation of capital) can be solved.

$$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha} = C_t + K_{t+1} - (1-\delta)K_t \implies K_{t+1} = Y_t - C_t$$
$$K_{t+1} = s_t Y_t$$
$$C_t = (1-s_t)Y_t$$
$$r_t = \alpha Y_t / K_t - \delta = \alpha Y_t / K_t - 1$$
$$1 + r_t = \alpha Y_t / K_t$$

- Together with the Euler equation this implies the non-linear stochastic equation(c.f. Ch2b-14)

$$\frac{s_t}{1-s_t} = e^{-(\rho-n)} \alpha E_t \frac{1}{1-s_{t+1}}$$

- Conjecture (by transversality conditions and feasibility of solutions): constant solution $s_t = s_{t+1} = \hat{s} \implies \hat{s} = \alpha e^{n-\rho}$
- Labor supply: $w_t / C_t = b/(N_t - L_t)$
- Labor demand: $w_t = (1-\alpha) Y_t / L_t$
- $$\frac{L_t}{N_t} = \frac{1-\alpha}{1-\alpha+b(1-\hat{s})} = 1 - \hat{u}$$

  where $\hat{u}$ is "voluntary unemployment"
- Both the optimal saving rate and per-capita labor supply (and therefore unemployment rate) are constant, which is a result of the assumption (Cobb-Douglas production, logarithmic utility, full depreciation of capital)

- Dynamics of GDP (Ch2b-16, notes): Second order stochastic difference equation (D_1, D_2 constant)

$$y_t = (\alpha + \rho_A)y_{t-1} - \alpha\rho_A y_{t-2} + (1-\alpha)\epsilon_t + D_1 + D_2 t$$

- **Trend component** $\bar{y}_t$ is defined as linear solution of

$$\bar{y}_t = (\alpha + \rho_A)\bar{y}_{t-1} - \alpha\rho_A \bar{y}_{t-2} + D_1 + D_2 t$$

  where linearity reflects balanced growth path property.

- The only linear solution is $\bar{y}_t = F + (n+g)t$ with F constant, and therefore

$$\bar{Y}_t = e^{\bar{y}_t t} = e^F e^{(n+g)t}$$

- The cycle component is then given by $\tilde{y}_t = y_t - \bar{y}_t$.
- The cycle component is an AR(2) process:

$$\tilde{y}_t = y_t - \bar{y}_t = (\alpha + \rho_A)\tilde{y}_{t-1} - \alpha\rho_A \tilde{y}_{t-2} + (1-\alpha)\epsilon_t$$

  with $V(\tilde{y}_t) \to +\infty$ as $\rho_A \to 1$ ("random walk").

- With ρ_A close to one, the effects of shocks last for several years (impulse response analysis) $\Longrightarrow$ ρ_A is of crucial importance for the structure of the business cycle.
- **Summary**: The special case provides some insights (Ch.2b-22), but leads to unrealistic results, because the saving rate is constant (consumption, investment and production would have some volatility, contrary to observations) and the unemployment rate is constant (observed to be countercyclical).

(c) The general case

- Typical procedure to solve general DSGE models:
  - Compute balanced growth path of deterministic model (trend components): $\bar{Y}_t, \bar{C}_t, \bar{L}_t$ etc.
  - Rewrite equilibrium conditions in logarithmic variables
  - Linearize equilibrium conditions around the trend component, yielding a system of equations which are linear in the cycle components: $\tilde{y}_t = y_t - \bar{y}_t$ etc.
  - To solve the linear system, make a linear guess: $\tilde{c}_t = \beta_{ck}\tilde{k}_t + \beta_{ca}\tilde{a}_t$
  - Determine βs (reaction coefficients) by substituting the linear guess into the log-linearized equilibrium conditions.
- Deterministic model: $N_t = N_0 e^{nt}, A_t = A_0 e^{gt}$
- Balanced growth path: $\bar{K}_t = K_0 e^{(n+g)t}, \bar{C}_t = C_0 e^{(n+g)t}, \bar{L}_t = (1-\bar{u})N_0 e^{nt}$
- Substituting into the equilibrium conditions yields 5 equations, which can be solved for 5 unknowns $\bar{u}, r_0, w_0, C_0, K_0$ (Ch.2c-24)
-
- Log-linearization Ch.2c-25, notes
- In the standard RBC model, equilibria are Pareto efficient.
- Business cycles are the reaction of the economy to exogenous (real) shocks.
- Criticism: For realistic results, one needs strong productivity shocks and a high elasticity of labor supply. Monetary shocks do not play any role.
- Under imperfect competition, distortionary taxation and external effects, equilibria are no longer Pareto efficient and the planner's solution differs from the equilibrium solution.

3. New Keynesian theories

(a) Review of traditional Keynesian analysis

- Main assumptions: Nominal rigidity (prices do not adjust; constant); real and monetary demand and supply shocks are the main causes of business cycles.
- Modelling strategy: Direct specification of relationships between variables (no optimization, microfoundation is **not part** of the model); expectations are usually not rational (i.e., model-consistent).
- Simplicity and transparency, but welfare analysis is not possible, and no microfoundation.
- Aggregate demand for goods and services in a closed economy can be decomposed in the following three components: C, I, G
- Aggregate supply of goods and services is given by GDP: Y
- If goods market clears, $Y = C + I + G$ (achieved through quantity adjustment, i.e., firms satisfy the demand and goods prices are fixed).
- Private consumption $C = C(Y - T, r, \epsilon)$, where $Y - T$ is disposable income, $0 < C_1 < 1, C_3 > 0$, r real interest rate, ϵ "state of confidence" (sign of C_2 depends on income- and substitution effects).
- Private investment $I = I(Y, r, \epsilon), I_1 > 0, I_2 < 0, I_3 > 0$.
- Government consumption is entirely financed by taxes (balanced budget): $G = T$ (exogenously given).
- We assume $C_1 + I_1 < 1$ and $C_2 + I_2 < 0$
- Assume trend components $\bar{Y}, \bar{G}, \bar{\epsilon}$ given. Then $\bar{r}$ is the solution of

$$\bar{Y} = C(\bar{Y} - \bar{G}, \bar{r}, \bar{\epsilon}) + I(\bar{Y}, \bar{r}, \bar{\epsilon}) + \bar{G}$$

- Taking total differentials (Ch.3a-32, notes) yields the **IS-curve**

$$\tilde{y}_t = -\alpha_r \tilde{r} + \alpha_g \tilde{g} + \tilde{v}$$

  where parameters $\alpha_r > 0, \alpha_g > 0$ and $\tilde{v}$ represents a shock to the IS-curve.
- Demand for real money balance is

$$M^D/P = L(Y, i)$$

  where M^D/P real money demand, M nominal money supply, M^D nominal money demand, i nominal interest rate, L money demand (liquidity preference) and $L_1 > 0, L_2 < 0$, because you forgo interest income.
- Assumption: Bonds yield interest, but cannot be used in transactions. Money does not yield interest, but can be used in transactions.
- Equilibrium on money and bonds market required that real money supply M/P equals real money demand M^D/P, i.e., the **LM-curve**

$$M/P = L(Y, i)$$

- Typical assumption: $L(Y, i) - kYe^{-\beta i}$, where k and β are positive parameters.
- If $\beta = 0$, this simplifies to $M = kPY$, i.e., nominal GDP is proportional to nominal money supply, i.e., to carry out all transactions, each unit of money must be used $1/k$ times (transaction velocity of money).
- $r = i - \pi^e$ (expected, consistent inflation; but NOT necessarily "rational expectation" - TODO what is the difference?), i.e, prices move, but slowly (Fisher equation).

- Taking logarithm yields $m - p = y - \beta i + \kappa$
- LM-curve may include (monetary) shocks, e.g., transaction velocity of money may be stochastic.
- Along the trend, $\bar{m} - \bar{p} = \bar{y} - \beta(\bar{r} + \pi^e) + \bar{\kappa}$, which for the cycle component yields

$$\tilde{m} - \tilde{p} = \tilde{y} - \beta\tilde{r} + \tilde{\kappa}$$

- Theoretical arguments for positive relation between price level and output level (aggregate supply curve), based on the way firms set prices and wages (expectations about future prices).
- A typical form of the aggregate supply curve might look like

$$p = p^e + \delta\tilde{y} + \tilde{\sigma}$$

  where p^e is the expected price level, $\delta > 0$ constant, $\tilde{\sigma}$ a supply shock.
- If expectations are formed in an adaptive way, e.g., $p^e = \bar{p} + \eta\tilde{p}_-$ with $\eta \in (0,1)$ and $\tilde{p}_-$ denoting last period's deviation of the price level from trend, this can be written as (Ch.3a-35)

$$\tilde{p} = \eta\tilde{p}_- + \delta\tilde{y} + \tilde{\sigma}$$

- **A simple (traditional) Keynesian model**: aggregate demand block (IS-curve plus LM-curve)

$$\tilde{y}_t = -\alpha_r\tilde{r}_t + \alpha_g\tilde{g}_t + \tilde{v}_t$$
$$\tilde{m}_t - \tilde{p}_t = \tilde{y}_t - \beta\tilde{r}_t + \tilde{\kappa}_t$$

  and the aggregate supply curve

$$\tilde{p}_t = \eta\tilde{p}_{t-1} + \delta\tilde{y}_t + \tilde{\sigma}_t$$

  which under some conditions can be solved for endogenous variables $(\tilde{y}_t)_{t=0}^{+\infty}, (\tilde{r}_t)_{t=0}^{+\infty}$ and $(\tilde{p}_t)_{t=0}^{+\infty}$, which shows how these variables depend on the exogenous shocks and policy instruments.
- Criticism on traditional Keynesian models
  - The central assumption of **price stickiness** is not well captured by the model.
  - Money supply is treated as the central bank's instrument variables, but most nowadays most central banks target the (short term) nominal interest rate.
  - The consumption demand function and the investment demand function are not microfounded.
  - Expectations are not treated in a satisfactory way.
  - **Lucas critique**: Changes in policy affect behavior of the agents as well as their expectations, which invalidates the estimates from econometric models.

(b) The basic new Keynesian model

- Methodology as in RBC models (DSGE, rational expectations, microfoundation), but specific assumption about nominal rigidity.
- Reasons for nominal rigidity:
  - Nominal wages and/or goods prices are fixed by contracts with positive duration.
  - Price adjustment costs (**menu costs**).
  - Lags in the transmission of information, so not all prices reflect the most recent information.
- Prices are set by economic agents (not everyone is a price takes) $\Longrightarrow$ **imperfect information** $\Longrightarrow$ equilibria are not Pareto efficient (solution differs from the solution of the planning model).
- Infinitely-lived households who supply labor to firms and use the resulting income to buy a (single) consumption good.

- Income that is not used for consumption is saved, i.e., holding nominal one-period government bonds (or private credit).
- Final output (consumption good) is produced by **competitive firms** from a range of intermediated goods.
- Intermediate goods are produced from labor. Each intermediated good producer is a monopolist who sets a profit maximizing price (**monopolistic competition**).
- In any given period, only a certain fraction of intermediate goods producing firms can reset their prices (**Calvo pricing**; unrealistic assumption, but produces good results).
- The central bank sets the nominal interest rate.
- **Simplifying assumptions**: No capital, and hence, no investment.
- Money as a **unit of account**, but not as an asset (cashless limit economy).
- Productivity shocks and monetary policy shocks are the only source of uncertainty.
- No population growth.
- $$U(C, L) = \frac{C^{1-\sigma}}{1-\sigma} - \frac{bL^{1+\phi}}{1+\phi}$$

  where $\sigma > 0$ (CES), $b > 0$ (weight parameter) and $\phi > 0$
- Unit interval of identical households (per-household variables coincide with aggregate variables).
- Real consumption in t is denoted by C_t, labor supply L_t.
- The representative household maximizes

  $$E_0 \sum_{t=0}^{+\infty} \beta^t U(C_t, L_t)$$

  where $\beta = e^{-\rho}$ and $\rho > 0$ is the time-preference rate.
- Flow budget-constraint of the household:

  $$P_t C_t + B_t \leq B_{t-1}(1 + i_{t-1}) + W_t L_t + D_t$$

  where P_t is the price of one unit of consumption, B_t bond holdings, i_t nominal interest rate, W_t nominal wage rate, and D_t nominal profits (imperfect competition!).
- Household chooses sequences of consumption rates, labor supplies and bond holdings.
- Optimality conditions (Ch.3b-42, notes; proof Ch.3b-43) yield:

  $$\bar{b} + \phi l_t = w_t - p_t - \sigma c_t$$
  $$c_t = E_t(c_{t+1}) - (1/\sigma)[i_t - E_t(\pi_{t+1}) - \rho]$$

  where $\pi_t = p_t - p_{t-1}$ is the rate of inflation from $t - 1$ to t.
- **Goods market clearing** (New Keynesian IS-curve, i.e., $Y_t = C_t$: All consumption goods produced in t must be consumed in t (no capital, no investment), i.e., $y_t = c_t$. Together with the Euler equation, this yields the **New Keynesian IS-curve**:

  $$y_t = E_t(y_{t+1}) - (1/\sigma)[i_t - E_t(\pi_{t+1}) - \rho]$$

- According to the Fisher identity, this is equal to

$$y_t = E_t(y_{t+1}) - (1/\sigma)(r_t - \rho)$$

i.e., current output depends positively on expected future output, and negatively on the real interest rate (**difference to the traditional IS-curve**: there are now expectations involved!).

- The final good (consumption good) is produced from a range $[0, 1]$ of intermediate goods according to the CES production function (Dixit & Stiglitz model of monopolistic competition)

$$Y_t = \left[\int_0^1 Y_t(i)^{(\eta-1)/\eta} di\right]^{\eta/(\eta-1)}$$

where $\eta > 1$ describes the **elasticity of substitution** between the various inputs (Cobb-Douglas is a special case with $\eta = 1$). Constant returns to scale.

- Final good firms take the intermediate goods price $\{P_t(i)|i \in [0, 1]\}$ as given, produce final output at minimal cost, and sell it at a price equal to marginal cost. The price P_t of one unit of the consumption good is therefore the minimum of

$$\int_0^1 P_t(i)Y_t(i)di \text{ subject to } 1 = \left[\int_0^1 Y_t(i)^{(\eta-1)/\eta} di\right]^{\eta/(\eta-1)}$$

where the left expression are the total production costs.

- Therefore,

$$P_t = \left[\int_0^1 P_t(i)^{1-\eta} di\right]^{1/(1-\eta)} \text{ and } Y_t(i) = \left[\frac{P_t(i)}{P_t}\right]^{-\eta} Y_t$$

where P_t is the Dixit/Stiglitz price index.

- Each good $i \in [0, 1]$ is produced by a monopolist, facing demand $Y_t(i) = [P_t(i)/P_t]^{-\eta} Y_t$ and having production function $Y_t(i) = A_t L_t(i)$, where A_t is the (random) productivity in t and $L_t(i)$ the employment in firm i.

- Profit of firm i is

$$D_t(P_t(i)) = P_t(i)Y_t(i) - W_t L_t(i) = [P_t(i)/P_t]^{-\eta}[P_t(i) - W_t/A_t]Y_t$$

where W_t/A_t is the (nominal) marginal cost of intermediate goods production in t. If the firm could choose its price optimally in every period, it would choose

$$P_t(i) = \frac{\eta}{\eta - 1}(W_t/A_t)$$

i.e., a fixed factor times its nominal marginal cost (constant price-markup).

- Under flexible prices (no nominal rigidity), the logarithm of real marginal cost denoted by μ is

$$\mu = w_t - a_t - p_t = -\ln[\eta/(\eta - 1)]$$

i.e., all intermediate goods producing firms choose the same price, i.e., $p_t = p_t(i)$ for all i.

- Optimal labor supply and goods market clearing gives $\bar{b} + \phi l_t = w_t - p_t - \sigma y_t$, optimal price setting gives $\mu = w_t - a_t - p_t$, and the production function yields $y_t = a_t + l_t$.

- Combining yields

$$y_t^n = \frac{1 + \phi}{\sigma + \phi} a_t + \frac{\mu + \bar{b}}{\sigma + \phi}$$

as the output in the **flexible price equilibrium** (natural level of output).

- **Calvo pricing**: In every period, only a randomly chosen fraction $1-\theta$ of firms may reset their prices. The remaining fraction θ must keep the price from the previous period.
- A firm that resets its price in t chooses P_t^* in oder to maximize

$$\sum_{s=0}^{+\infty} \theta^s E_t[Q_{t,t+s} D_{t+s}(P_t^*)]$$

  where $Q_{t,t+s} = \beta^s (C_{t+s}/C_t)^{-\sigma}(P_t/P_{t+s})$ is the stochastic discount factor, θ^s is the probability that the price P_t^* chosen in t is still valid in $t+s$.
- The stochastic discount factor reflects time-preference β^s, marginal rate of substitution of consumption $(C_{t+s}/C_t)^{-\sigma}$ and the relative price of consumption P_t/P_{t+s}.
- Close to a steady state with zero inflation, log-linearization yields (Ch.3b-49, notes)

$$p_t^* = \ln[\eta/(\eta-1)] + (1-\beta\theta)\sum_{s=0}^{+\infty}(\beta\theta)^s E_t(w_{t+s} - a_{t+s})$$

  which shows that the optimal price is a fixed markup $\eta/(\eta-1)$ times a weighted average of expected future marginal costs, whereby the weights reflect time-preference and the probability of the price remaining effective.

  This can also be written as

$$p_t^* = (1-\theta\beta)(w_t - a_t - \mu) + \theta\beta E_t(p_{t+1}^*)$$

- **New Keynesian Phillips curve**: All firms that set a new price in period t choose the same price. Therefore

$$P_t = \left[\theta P_{t-1}^{1-\eta} + (1-\theta)(P_t^*)^{1-\eta}\right]^{1/(1-\eta)}$$

  which upon log-linearizing around a steady state with zero inflation yields

$$\pi_t = (1-\theta)(p_t^* - p_{t-1})$$

  or combined

$$\pi_t = \beta E_t \pi_{t+1} + \lambda(\mu_t - \mu)$$

  where λ constant: the New Keynesian Phillips curve is also forward-looking due to the forward-looking behavior of intermediate goods firms (Ch.3b-50).
- Analogously to the calculation of the flexible price equilibrium we get

$$y_t = \frac{1+\phi}{\sigma+\phi} a_t + \frac{\mu_t - \bar{b}}{\sigma+\phi}$$

  and therefore

$$\mu_t - \mu = (\sigma+\phi)x_t$$

  where $x_t = y_t - y_t^n$ is the **output gap**.
- Natural rate of interest: $r_t^n = \rho + \sigma[E_t(y_{t+1}^n) - y_t^n]$
- $\pi_t = \beta E_t(\pi_{t+1}) + \kappa x_t$, where $\kappa = \lambda(\sigma+\phi)$ (Ch.3b-51).
- The IS-curve and the Phillips curve together form the non-policy block of the basic New Keynesian model, involving two endogenous variables (output gap and inflation), an exogenous disturbance term (natural rate of interest) and the nominal interest rate i_t.

- Central banks use short term interest instead of the nominal money growth rate as instrument variable.
- **Taylor** rule (1993): $i_h = \rho + h_\pi \pi_t + h_x x_t + \delta_t$, where h_π and h_x are non-negative parameters (e.g. for USA, $h_\pi \approx 1.5, h_x \approx 0.5$) and δ_t represents a stochastic monetary policy shock.
- **New Keynesian model** consists of the New Keynesian IS-curve, New Keynesian Phillips curve and the Taylor rule.
- Combination of the three (Ch.3b-54; notes) yields a "expectational difference equation", which has a unique stationary solution if and only if the determinacy condition holds.
- Central banks choose h_π, h_x s.t. the condition holds. If $h_x = 0, h_\pi > 1$ is necessary and sufficient for the condition to hold. $h_\pi > 1$ is called the **Taylor principle**, i.e., the central bank should react "sufficiently strongly" to deviations of inflation and output gap from their target levels.

(c) Effects of shocks and optimal policy in the New Keynesian model

- **Effects of monetary policy shocks**: Suppose δ_t is an AR(1) process and there are no other shocks, i.e., $r_t^n = \rho$ for all $t \geq 0$.
  - Guess and verify: $x_t = G_{x\delta}\delta_t, \pi_t = G_{\pi\delta}\delta_t$ (Ch.3c-56; notes).
  - As long as the determinacy condition is satisfied, a single contractionary monetary policy shock persistently reduces inflation and the output gap. As natural output is unaffected by the shock, output itself is also persistently reduced.
  - A strong response by the central bank to deviations of inflation and the output gap from their targets helps to stabilize inflation and the output gap simultaneously (there is **no trade-off**).
  - The policy parameters h_π, h_x affect the long-run variances of inflation and the output in the same way. There is **no trade-off** between output stabilization and inflation stabilization (Ch.3c-57).
- **Effects of technology shocks**: The productivity shock a_t is an AR(1) process, and $\delta_t = 0$ for all $t \geq 0$. Thus, $\tilde{r}_t^n = r_t^n - \rho$.
  - Guess and verify $x_t = G_{xa}\tilde{r}_t^n, \pi_t = G_{\pi a}\tilde{r}_t^n$.
  - Similar results to monetary policy shock, i.e., a single positive productivity shock persistently reduces inflation and the output gap (**not** the output, but the output gap!; output typically increases).
  - Employment is given by $l_t = y_t - a_t$ (TODO: where is k_t??). Taking derivatives shows that for small preference rate σ, the response is positive; for large σ (in particular $\sigma = 1$ for logarithmic preferences) the response is negative, which is in line with empirical evidence.
- Both monetary and policy shocks influence only the IS-curve, but not the Phillips curve.
- By allowing exogenous changes in the price markups or fluctuations in labor income taxes, the Ney Keynesian Phillips curve takes the form

$$\pi_t = \beta E_t(\pi_{t+1}) + \kappa x_t + \zeta_t$$

  where ζ_t denotes the **cost-push shock** (e.g., oil price shock).
- **Cost-push shock**: ζ_t as AR(1) process, and $\delta_t = \tilde{r}_t^n = 0$ for all $t \geq 0$.
  - Guess and verify as before: $x_t = G_{x\zeta}\zeta_t, \pi_t = G_{\pi\zeta}\zeta_t$. Difference to before: the reaction coefficients now also appear in the numerator (Ch.3c-60)!
  - None of the two policy parameters can be used to reduce the variability of both the output gap and the inflation rate. There is a **trade-off between stabilization of the output gab and stabilization of inflation**.

- **Social cost of business cycles**: The goal of stabilization policy is to minimize

$$Z = \sigma_\pi^2 + \omega \sigma_x^2$$

where σ^2 denotes the respective long-run (asymptotic) variance, and the parameter ω the weight of stable output relative to stable inflation.

- **Reasons for output stabilization**: Less fluctuations of income and consumption (which risk averse households don't like, i.e., causes a loss of welfare); imperfect capital and insurance markets don't enable households to smooth their consumptions; fluctuation of employment cause additional social costs.

- **Reasons for inflation stabilization**: Price fluctuations makes intertemporal decisions difficult.

- The inflation target should be low, because high inflation causes shoeleather costs, menu costs and relative price distortions (Calve pricing).

- The inflation target should be positive, because wage and price rigidity are asymmetric. A positive rate of inflation facilitates real interest rate cuts for the central bank (because the nominal interest rate cannot become negative).

- The optimal value of h_π and h_x depends on ω and the relative frequency and size of the various types of shocks.